tive groups.

The reader who desires a thorough working knowledge of this subject could very profitably work over this field and compare his methods and results with those that we shall give.

THE GOLDEN SECTION.

By EMMA C. ACKERMANN, Instructor in Mathematics, Michigan State Normal School.

In the number for November 1892, of *Lehrproben und Lehrgange aus der Praxis der Gymnasien und Realschulen*, there appeared an article by Prof. Dr. O. Willman, entitled Der goldene Schnitt abs ein Thema des mathematischen Unterrichts. The article is interesting not alone to students of geometry, but to all who are at all concerned with the question of concentration, a question which is at present creating for itself an active interest among all educators. The article is a resume of a work on the golden section by F. C. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur, und Kunst, Augsburg, Huttler, 1885. The statements here presented are intended as a summary of the article.

It is very necessary that the connection between instruction in mathematics and in the remaining courses of study should be carefully considered because the subject of mathematics is an abstract one and according to its nature tends towards isolation.

To assist in bringing about this connection, there should be prepared mathematical problems and exercises which will show the application of mathematics to technics and to observations in nature on the one hand, and on the other furnish an insight into the history of mathematics, by means of which historical and classical instruction can be connected with the mathematical. A subject which meets these demands and is at the same time well adapted for purely mathematical instruction is the theme of the golden section, a theme which does not appear in a complete form in our modern text-books.

The simplest division of any magnitude, involving the fewest conditions is the division into two equal parts. Calling a line so divided, S, the parts p, we have $S=2p$, $p=\frac{S}{2}$, $\frac{S}{p}=2$, $\frac{p}{S}=\frac{1}{2}$, $\frac{p}{p}=1$. Contrasted to one case of division into two equal parts stands an infinite number of divisions into two unequal parts; and the ratio of the smaller (m) to the larger (M), $\frac{M}{m}$, or the ratio of one of the parts to the whole, $\frac{m}{S}$ or $\frac{m}{m+M}$ and $\frac{M}{S}$ or $\frac{M}{m+M}$ can be expressed by many different numbers. In one case only is there no need of figures to determine the ratio of the parts to the whole; and that is when

$\frac{m}{M}=\frac{M}{m+M}$. Such a division constitutes the golden section.

From the proportion $m:M=M:m+M$, we have $mM+m^2=M^2$ or $(M+m)(M-m)=mM$; or in the golden section, the sum of the parts multiplied by their difference equals their product. It also follows that the greater part is the geometric mean between the smaller part and the whole. Let $S=m+M$; then $S-m=\sqrt{Sm}$; or the difference between the whole and the smaller part is the geometric mean of those two parts. Also both parts form with the whole a continued proportion, $m:M=M:m+M$ or $m:M:m+M$, distinguished from all other proportions by the fact that the third quantity is at the same time the sum of the other two. From $\frac{M}{m}=\frac{M+m}{M}$, we have $\frac{M}{m}=1+\frac{m}{M}$ or $\frac{m}{M}=\frac{M}{m}-1$; that is, the quotient of the two parts is greater or less than its reciprocal by unity. Call the ratio $\frac{M}{m}$, e, then we have the equation $e=1+\frac{1}{e}$, solving $e=\frac{\sqrt{5}+1}{2}=1.61803+$.

Of the three elements of the golden section, each two can be expressed by the third. The simplest is that of m and S by means of M; $m=\frac{\sqrt{5}-1}{2}M$; $S=\frac{\sqrt{5}+1}{2}M$. Also $m=\frac{3-\sqrt{5}}{2}S$ and $M=\frac{\sqrt{5}-1}{2}S$, that is, the major part becomes the minor, when the whole is considered the major. If m is the base, $M=\frac{\sqrt{5}+1}{2}m$, $S=\frac{\sqrt{5}+3}{2}m$.

For the construction and consideration of the golden section, the number 5, which appears in the value of e is very suggestive: $5=1+4$, and therefore can be expressed by the Pythagorean theorem; 5 is the area of the square on the hypotenuse, if $1+4$ are the squares on the other two sides. The sides themselves are 1 and 2, and the hypotenuse, $\sqrt{5}$. Half of $\sqrt{5}$ increased by $\frac{1}{2}$ of unity will then express the value of e, and diminished by $\frac{1}{2}$ of unity, its reciprocal. A line AB is divided into medial section therefore, if in the right triangle ABC, with right angle ABC, we make $BC=\frac{1}{2}AB$, draw AC, lay off $CD=BC$, and then lay off the remainder AD or AB as AE; then AB is divided into medial section.

The number 5 may be used in another way to illustrate the ratio of m to M. An isosceles triangle with angles $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{2\pi}{5}$ is constructed. By bisecting one of the equal angles, we have a triangle similar to the first. Let the triangles be ABC and ABD respectively, angle B being $\frac{\pi}{5}$; then CB is divided in medial section, from principles of similar triangles. The triangle ABC is

middle part of a pentagon which is completed by placing two triangles congruent to triangle ABD on AB and BC. AD produced will then pass through an angle of the pentagon and CD becomes the smaller part and BD the larger part of a diagonal. Therefore in a regular polygon of five sides the smaller part of a diagonal cut off by a second diagonal forms with the side of the pentagon and the diagonal, the proportion of the golden section. The triangle is also an element of the regular decagon and will produce it if repeated ten times.

In the division of magnitudes into two equal parts, the whole may be considered as one of the parts repeated; so in the golden section, each one of the parts may be considered as the starting-point and the next as a repetition of it augmented or diminished. If we proceed from the minor part, the major is a repetition of the minor increased, and the sum of the two bears the same relation to the major, the ratio in each case being e. So if we proceed from the whole to the major, and from that to the minor. With this view of the case, there is no necessity for stopping with three elements, since this augmenting or diminishing repetition can evidently be carried on indefinitely. In this way the geometric proportion of the golden section becomes a geometric progression which from analogy is called the golden progression, the ratio being e or $\frac{1}{e}$. The golden progression differs from the other geometric series in this that each of its members is also the sum of the two preceding.

If its first term is a, then this progression has the form a, ae, ae^2, ae^3, $ae^4 \ldots ae^n$. or a, ae, $a+ae$, $a+2ae$, $2a+3ae$, $3a+5ae \ldots$

Then $a^2=a(1+e)$; $ae^3=a+2ae$; $ae^4=a(2+3e)$, etc. If $a=1$, then since $e=\frac{\sqrt{5}+1}{2}=1.61803$, the series is: 1, 1.61803+, 2.16803+, 4.23607+, 6.185410+, 11.09017+, 17.94427+, 29.03444+, etc. If we should tentatively place e also equal to 1, that is, 0.61803 too small, then as a geometric progression simply, the series remain stationary. But using the other property of the golden progression, the series becomes 1, 1, 2, 3, 5, 8, 13, etc. The quotient of any two successive members is alternately smaller and larger than e, but as the series advances, the quotient approaches nearer to e, as at $\frac{233}{144}=1.618$. This shows that a series beginning with the smallest natural numbers and advancing according to the second condition above, forms in its continuation an approximation to the golden progression.

The golden progression can be represented graphically thus: A pentagon is drawn whose sides are F and diagonals D; the intersections of the diagonals determine a pentagon whose sides are f and whose angles are at a distance a from the vertices of the original polygon. Then f, a, F, D form a golden series. By making the original polygon the enclosed polygon of a larger pentagon, whose sides and diagonals are F' and D' respectively, we can continue the progression as an ascending series; or, by drawing polygons within the pentagon, as a decending series.

Another interesting figure might be given here. The line AB is divided in medial section at C, AC being the major part; perpendiculars BD,

CF, of length AC are drawn at B and C, and are divided at E and G respectively into medial section, BE and GC being the major parts. Then AB, AC $(=BD=CF)$, $BC(=BE=CG)$ and $DE(=FG)$ form a golden series. Connect F and A, G and A, E and A. Let AC be taken as a radius or unity; AC then represents the side of a hexagon; AF the side of a square; AG, of a regular pentagon, EB or CG, the side of a decagon; and AE, the side of a regular triangle, all of which is evident from the right angled triangle.

Another figure may be obtained by making m, M, $m+M$ the radii of concentric circles. Their areas are then πm^2, πM^2, $\pi(m+M)^2$, the area of the inner ring, r, is $\pi(M^2-m^2)$ and of the outer ring, R, $\pi[(M+m)^2-M^2]$. If $m=1$, then $M=e$, $m+M=e^2$, $(M+m)^2=e^4$, $M^2-m^2=e^2-1$, and $(M+m)^2-M^2=e^4-e^2=e^2(e^2-1)=e^3$. Then the following series arises:

$\pi=$area of inner circle, f;
$e\pi=$ " " " ring, r;
$e^2\pi=$ " " middle circle, F;
$e^3\pi=$ " " outer ring, R;
$e^4\pi=$ " " " circle, F'.

This series can be extended both as increasing and decreasing; the members with even exponents as e^6, e^8, e^{-2}, e^{-4} correspond to circles; those with odd exponents to rings.

Countless illustrations of the proportions of the golden section are found in nature and the works of man. The golden section follows closely upon bisection (the basis of symmetry) everywhere, and the forms which are based upon the proportions of the golden section though not so evident are more widely distributed than would appear at first thought. Whenever, in the products of art or manufacture, there is no equal division, (symmetry), the artist or workman unconsciously employs the proportions of the golden section. Irregular inequality and capricious division is disagreeable to both eye and hand; and the porportion of the golden section seem to be the only acceptable ones. Accordingly, the form of writing-paper, books, a page of the MATHEMATICAL MONTHLY, furniture, especially tables and chairs, doors, windows, dimensions of pictures, foundations and often the facades of buildings, all reveal these proportions.

This is true of not only modern art and technics, but also of the ancient. We find the same proportions in the pyramids of Cheops, in the temples at Karnak and at Ombos, in the Grecian temples, and many cathedrals.

In verses of poetry and in music, the same relation is found, and most abundantly in nature. In leaves, plants, lower animals, and man, these proportions have been verified.

The subject of the golden section is not discussed by Euclid; he had a knowledge of it and mentions it in his works, though not under this name. The name, though it has an ancient ring, is not found in ancient literature. Aristotle does not mention the subject, but it is claimed that in his philosophical reasoning, there rules the principle of the golden section; *i. e.*: the relation of the whole to the part and the parts to each other. His ideas were not carried

out by the ancient philosophers but they were the source of much of the speculation in mediaeval times, when mathematical and philosophical thought were closely allied. One writer, John Campanus of Novava, thought that the principiple of the golden section descended from the gods. Keplertcompared it to a precious stone, and called it *proportio divina*, but not *proportio* or *sectio aurea*. The latter name has originated since his time.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

(*Continued from the July-August Number.*)

The central-polar equation of the Cassinian Oval may be written

$$r^4-(2c^2\cos 2\theta)r^2=m^4-c^4\ldots\ldots(1).$$

$$\therefore\ \cos 2\theta=\frac{r^4-(m^4-c^4)}{2c^2r^2},\ \text{and}\ \sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[r^4-(m^4-c^4)]}{4c^4r^4}\right)}.$$

$$\therefore\ \mathsf{P}=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{4c^4r^4-[r^4-(m^4-c^4)]^2\}}}$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ldots\ldots(2).$$

Reducing (2) under the supposition that

$$r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi,$$

$$\mathsf{P}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{r}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}}\ldots\ldots(j),$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2(1-2\sin^2\phi)]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2\cos 2\phi]^{\frac{1}{4}}}\ldots\ldots(3).$$

Let $2\phi=\psi$, and make $2m^2c^2 / (m^4+c^4)=C$; then, after obvious transformations, (3) gives